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# ISOPERIMETRIC INVARIANTS FOR PRODUCT MARKOV CHAINS AND GRAPH PRODUCTS

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Bounds on some isoperimetric constants of the Cartesian product of Markov chains are obtained in terms of related isoperimetric quantities of the individual chains.

## 1. Introduction

Isoperimetric inequalities have long been of interest to geometers and analysts, and they have become central to some parts of probability theory. They have also proved useful in analyzing the mixing rates of Markov chains ([20], [12], [13], [1]). We wish in these notes to provide another example of this usefulness by deriving tight bounds for some isoperimetric constants of the Cartesian product of Markov chains. The impetus for the work presented here are the results obtained in [5], where dimension free bounds are obtained for the isoperimetric constants of product probability measures. Some of our results can thus be seen as the Markovian versions of those obtained in [5] and at times heavily rely on some of the ideas and techniques presented there. However important differences, linked mainly to the existence of many discrete gradients, occur in the Markovian setting.

Let us briefly present our main result: Let the triple $(X, K, \pi)$ be a Markov kernel $K$ with invariant measure $\pi$ on a finite state space $X$. For any $1 \leq$

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$p<+\infty$, let

$$h_p(K) = \min_{\substack{A\subset X \\ 0<\pi(A)<1}} \frac{\sum_{x\in A}\left(\sum_{y\in A^c} K(x,y)\right)^{1/p}\pi(x) + \sum_{x\in A^c}\left(\sum_{y\in A} K(x,y)\right)^{1/p}\pi(x)}{\min(\pi(A),\pi(A^c))}, \tag{1.1}$$

where $A^c$ is the complement of $A$ in $X$, and where $\pi(A)=\sum_{x\in A}\pi(x)$. For $p=+\infty$, we have the usual modification and

$$h_\infty(K) = \min_{\substack{A\subset X \\ 0<\pi(A)<1}} \frac{\pi\{x\in A : \sum_{y\in A^c} K(x,y) > 0\} + \pi\{x\in A^c : \sum_{y\in A} K(x,y) > 0\}}{\min(\pi(A),\pi(A^c))}. \tag{1.2}$$

For notational convenience, we write $h_p(K)$ rather than $h_p(K,\pi)$ and hope that the underlying $\pi$ is understood from the context. For $p=1$, $h_1(K)$ is just twice the familiar isoperimetric (or Cheeger's) constant (or conductance) of the Markov chain after which our general definition is modeled. *Throughout the text, when needed we assume that all the (isoperimetric) constants defined are positive. As easily verified, an irreducibility assumption on the Markov chain will ensure this fact.*

Now, for each $i=1,2,\dots,n$, let $(X_i,K_i,\pi_i)$ be a Markov kernel $K_i$ with invariant measure $\pi_i$ on a finite state space $X_i$. A standard way of considering the product of Markov kernels is the notion of Cartesian product (e.g., [1], [2], [11]) defined as follows: Let $X^n=\prod_{i=1}^n X_i$, and let

$$K^n(x,y) = \frac{1}{n}\sum_{i=1}^n \delta(x_1,y_1)\cdots\delta(x_{i-1},y_{i-1})K_i(x_i,y_i)\delta(x_{i+1},y_{i+1})\cdots\delta(x_n,y_n),$$

where as usual, $\delta(x_k,y_k)=1$, if $x_k=y_k$ and 0 otherwise. It is then easily verified that $\pi^n$, the invariant measure of $K^n$, is the product probability measure given by $\pi^n=\pi_1\otimes\cdots\otimes\pi_n$. As a particular case of a general result, we show:

**Theorem 1.1.** *Let $(K_i,\pi_i)$ be Markov chains on the finite state space $X_i$, $1\le i\le n$, and with Cartesian product $(X^n,K^n,\pi^n)$. Let $1\le p\le 2$, then*

$$h_p(K^n) \ge \frac{1}{4\sqrt{6}n^{1/p}}\min_{1\le i\le n} h_1(K_i). \tag{1.3}$$

Since trivially, for any $1 \le p \le 2$,

$$h_p(K^n) \le \frac{1}{n^{1/p}} \min_{1\le i\le n} h_p(K_i), \tag{1.4}$$

(1.3) is of best possible order in $n$. Although independent of $p$ and $n$, $1/4\sqrt{6}$ is not the optimal constant (see Concluding Remarks). There is a certain discrepancy between (1.3) and (1.4) in that the minima involve $h_1$ and $h_p$ respectively, and clearly, $h_p(K)$ is non-decreasing in $p \in [1, +\infty]$. If $K(x,y) \ge C(K) \ge C$, for some absolute constant $C$ and all $x, y \in X$ such that $K(x,y) > 0$, this discrepancy disappears and (changing the constant) one can replace in (1.3) $\min_{1\le i\le n} h_1(K_i)$ by $\min_{1\le i\le n} h_p(K_i)$. In general, however, this additional assumption on $K$ cannot be avoided (see Remark 4.4). For $p = 1$, the absolute constant can be further improved (again, see Concluding Remarks). This improved version has recently been used to study the mixing time of the so-called Wolff process in Statistical Physics (see [10]).

The eigenvalue information is hopeless in getting the above theorem, even in the simplest case of $p = 1$. Indeed, let $(X, K, \pi)$ be as above with moreover, $K$ reversible ($K(x,y)\pi(x) = K(y,x)\pi(y)$, for all $x, y \in X$). Also, let $I$ be the $\#X \times \#X$ identity matrix, where $\#$ denotes cardinality. Let the Laplacian of $K$ (viewed as a matrix) be defined as

$$L = L(K) = I - K.$$

Now, let $\lambda(L)$ be the *spectral gap* of the Laplacian, i.e., the second smallest eigenvalue of $L$. The following inequalities relating the spectral gap to $h_1(K)$ have been obtained, with minor variations in the constants by various authors (e.g., [2], [20], [18], [12], [1]):

$$\frac{h_1^2(K)}{8} \le \lambda(L(K)) \le h_1(K). \tag{1.5}$$

Now, given $n$ reversible Markov chains, $(K_1, \pi_1), (K_2, \pi_2), \ldots, (K_n, \pi_n)$ on finite sets $X_i$, it is not difficult to show ([2], [11], [17]) that for the product Laplacian $L(K^n)$,

$$\lambda(L(K^n)) = \frac{1}{n} \min_{1\le i\le n} \lambda(L(K_i)). \tag{1.6}$$

Combining (1.5) and (1.6) gives

$$h_1(K^n) \ge \lambda(L(K^n)) = \frac{1}{n} \min_{1\le i\le n} \lambda(L(K_i)) \ge \frac{1}{8n} \min_{1\le i\le n} h_1^2(K_i),$$

an inequality which is weaker than (1.3) for $h_1$ small.

As part of the work presented here, we briefly discuss how to derive analogous results for Cartesian products of graphs. Some standard graphs of interest are the discrete cube, the discrete torus, the grid graph, and the Hamming graph. The factor graphs from which the Cartesian product graphs are built, are respectively an edge, a cycle, a path, and a complete graph. In each case the exact computation, of various isoperimetric constants that we introduce, is easy in one dimension (i.e. for the factor graph), but a challenging task in $n$-dimensions. Our theorem provides a way of lower bounding the $n$-dimensional constants, whenever the one dimensional constants can be computed exactly or just estimated.

We also remind the reader that the paper deals with *edge*-isoperimetric inequalities, and not *vertex*-isoperimetric inequalities. The case $p=+\infty$ corresponds to vertex-isoperimetric inequalities (see [6], [17]). However, in general, information about vertex isoperimetry does not help in achieving *tight* edge-isoperimetric inequalities (e.g., for the special case of the grid-graph, the optimal edge as well as vertex isoperimetric inequalities are also known (see [7], [8], [9]).)

The paper is structured as follows. In Section 2 we define various discrete gradients and relate the total gradient to its partials. In Section 3, a result of independent interest and one which plays a key rôle in proving our results is the Markovian version of a generalization of Cheeger's inequality to Orlicz norms obtained in [5]. In Section 4, we prove (by induction) our main result and deduce a functional form inequality. In Section 5, we provide various corollaries bounding the *edge-isoperimetric number* of products of graphs. Finally in Section 6, we analyze our framework, and explain how some of our hypothesis can be removed and some of our results extended (continuous framework, arbitrary state space, ...).

## 2. The gradient in a discrete setting

Let $K$ be a Markov kernel on a finite state space $X$ with $\pi$ as the invariant measure. For any $1\le p<+\infty$, each $x\in X$, and any function $f:X\to\mathbb{R}$, let

$$\nabla_p^+ f(x) = \left(\sum_{y\in X}\left((f(x)-f(y))^+\right)^p K(x,y)\right)^{1/p},$$

where, as usual, for any real valued function $g$ on $X$, $g^+=\max(g,0)$. Let also $\nabla_\infty^+ f(x)=\sup_{\{y\in X:K(x,y)>0\}}(f(x)-f(y))^+$. Replacing above $(f(x)-f(y))^+$

by $(f(x)-f(y))^-$, where $g^-=\max(-g,0)$, we similarly define $\nabla_p^- f(x)$, $1\leq p\leq+\infty$, and so $\nabla_p^+ f=\nabla_p^-(-f)$. Let also

$$|\nabla_p f(x)| = \left(\sum_{y\in X} |f(x)-f(y)|^p K(x,y)\right)^{1/p},$$

$1\leq p<+\infty$, and $|\nabla_\infty f(x)|=\sup_{\{y\in X: K(x,y)>0\}}|f(x)-f(y)|$. Clearly,

$$|\nabla_p f(x)|^p = (\nabla_p^+ f(x))^p + (\nabla_p^- f(x))^p,\ p<+\infty,$$

with moreover

$$\max(\nabla_p^+ f(x), \nabla_p^- f(x)) \leq |\nabla_p f(x)| \leq (\nabla_p^+ f(x)) + (\nabla_p^- f(x)),$$

$$\Big|\nabla_p|f|(x)\Big| \leq |\nabla_p f(x)|,$$

$1\leq p\leq+\infty$. Hölder's inequality also shows that $|\nabla_p f(x)|$, $\nabla_p^+ f(x)$ and $\nabla_p^- f(x)$ are non-decreasing in $p$. Moreover, for any $x\in X$, $p\leq q\leq+\infty$,

$$\left(\inf_{y\in X: K(x,y)>0} K(x,y)\right)^{1/p-1/q} \nabla_q^+ f(x) \leq \nabla_p^+ f(x), \tag{2.1}$$

and so $p$ does not play any significant rôle whenever $K(x,y)$ is uniformly bounded below.

As readily checked, for any $A\subset X$,

$$\nabla_p^+ \mathbf{1}_A(x) = \mathbf{1}_A(x)\nabla_p^+\mathbf{1}_A(x) = \left(\sum_{y\in A^c} K(x,y)\right)^{1/p},\ \text{for } x\in A,$$

$$\nabla_p^- \mathbf{1}_A(x) = \mathbf{1}_{A^c}(x)\nabla_p^-\mathbf{1}_A(x) = \left(\sum_{y\in A} K(x,y)\right)^{1/p},\ \text{for } x\in A^c,$$

and similarly for $p=+\infty$; and so $\nabla_p^+\mathbf{1}_A = \nabla_p^-\mathbf{1}_{A^c}$. Thus, if $E_\pi$ denote expectation with respect to the stationary distribution $\pi$, we have

$$E_\pi|\nabla_p\mathbf{1}_A| = E_\pi\nabla_p^+\mathbf{1}_A + E_\pi\nabla_p^-\mathbf{1}_A,$$

$1\leq p\leq+\infty$.

Hence, our isoperimetric constants can be rewritten as

$$h_p(K) = \min_{\substack{A\subset X\\ 0<\pi(A)<1}} \frac{E_\pi|\nabla_p\mathbf{1}_A|}{\min(E_\pi\mathbf{1}_A, E_\pi\mathbf{1}_{A^c})} = \min_{\substack{A\subset X\\ 0<\pi(A)\leq 1/2}} \frac{E_\pi|\nabla_p\mathbf{1}_A|}{E_\pi\mathbf{1}_A},$$

$1 \le p \le +\infty$. In a similar fashion we define

$$h_p^+(K) = \min_{\substack{A\subset X\\ \pi(A)\le 1/2}} \frac{E_\pi \nabla_p^+ \mathbf{1}_A}{E_\pi \mathbf{1}_A},$$

and replacing above $\nabla_p^+$ by $\nabla_p^-$ gives $h_p^-$, $1 \le p \le +\infty$. For $p=1$, under reversibility, it is easy to see that $h_1 = 2h_1^+ = 2h_1^-$. (Actually, since $\pi K = \pi$, it follows that for any $f: X \to \mathbb{R}$, $E_\pi(Kf) = E_\pi f$, and thus $E_\pi \nabla_1^+ f = E_\pi \nabla_1^- f$. Hence, $h_1^+ = h_1^- = h_1/2$, without assuming reversibility). But for $p>1$, $h_p^-$ and $h_p^+$ can be significantly different quantities (see [25] or Remark 3.5). Note finally that for $1 \le p \le q$, $h_q^{+q} \le h_p^{+p}$.

Our definition of the gradients was motivated by various reasons: To unify in a single framework, edge and vertex isoperimetric languages; and thus to recover both conductance and expansion. To bypass the notion of Dirichlet form. To recover a notion of surface measure given on the discrete cube in [25] (see also [3]). In the notation of these authors, we have $Mf = \nabla_2^+ f$, and $\sqrt{h_A} = \nabla_2^+ \mathbf{1}_A$, $A \subset X$.

Let $X^n = X_1 \times X_2 \times \cdots \times X_n$, be the Cartesian product of $n$ state spaces, and let $K^n$ and $\pi^n$ be defined as in the previous section. For $x = (x_1, x_2, \ldots, x_n) \in X^n$, we write $x^{(n-1)}$ to denote $(x_1, x_2, \ldots, x_{n-1})$. Further, $\nabla_{p,x_n}^{\pm} f$, refers to the “positive” or “negative” one-dimensional gradient with respect to the coordinate $x_n$, and $\nabla_{p,x^{(n-1)}}^{\pm} f$ refers to the “positive” or “negative” $(n-1)$-dimensional gradient. They are respectively defined as:

$$\nabla_{p,x_n}^{\pm} f(x) = \left( \sum_{y_n \in X_n} \left( \left( f(x) - f(x^{(n-1)}, y_n) \right)^{\pm} \right)^p K_n(x_n, y_n) \right)^{1/p},$$

$$\nabla_{p,x^{(n-1)}}^{\pm} f(x) = \left( \sum_{y^{(n-1)} \in X^{n-1}} \left( \left( f(x) - f(y^{(n-1)}, x_n) \right)^{\pm} \right)^p K^{n-1}(x^{(n-1)}, y^{(n-1)}) \right)^{1/p},$$

and similarly for $|\nabla_{p,x_n} f(x)|$ and $|\nabla_{p,x^{(n-1)}} f(x)|$. We can now prove a key property of the discrete gradients.

**Lemma 2.1.** *Let $1 \le p < +\infty$ and let $1 \le q < +\infty$. Then, for $x \in X^n$,*

$$n^{q/p} \left( \nabla_p^+ f(x) \right)^q =$$

$$= \left[ (n-1) \left( \nabla_{p,x^{(n-1)}}^+ f(x) \right)^p + \left( \nabla_{p,x_n}^+ f(x) \right)^p \right]^{q/p} \tag{2.2}$$

$$\geq \min\left(2^{-1+q/p},1\right)\left[(n-1)^{q/p}\left(\nabla^{+}_{p,x^{(n-1)}}f(x)\right)^{q}+\left(\nabla^{+}_{p,x_n}f(x)\right)^{q}\right], \tag{2.3}$$

*and*

$$\nabla^{+}_{\infty}f(x)=\max_{1\leq i\leq n}\left(\nabla^{+}_{\infty,x_i}f(x)\right). \tag{2.4}$$

*Moreover $\nabla^{+}_{p}f$ can be replaced by either $\nabla^{-}_{p}f$ or $|\nabla_p f|$ in the above results.*

**Proof.** We only prove the results for $\nabla^{+}_{p}$, $p<+\infty$, the other cases being similar. Let $x=(x_1,\dots,x_n)\in X^n$, then

$$\left(\nabla^{+}_{p}f(x)\right)^{q}=\left[\sum_{y\in X^n}\left[(f(x)-f(y))^{+}\right]^{p}K^{n}(x,y)\right]^{q/p}.$$

Then,

$$\begin{aligned}
&K^n(x,y)=K^n((x_1,\dots,x_n),(y_1,\dots,y_n))\\
&=\frac{1}{n}\sum_{i=1}^{n}\left(K_i(x_i,y_i)\prod_{\substack{j=1\\ j\neq i}}^{n}\delta(x_j,y_j)\right)\\
&=\frac{1}{n}\left[\sum_{i=1}^{n-1}K_i(x_i,y_i)\prod_{\substack{j=1\\ j\neq i}}^{n}\delta(x_j,y_j)+K_n(x_n,y_n)\prod_{j=1}^{n-1}\delta(x_j,y_j)\right]\\
&=\frac{1}{n}\left[(n-1)K^{n-1}(x^{(n-1)},y^{(n-1)})\delta(x_n,y_n)+K_n(x_n,y_n)\prod_{j=1}^{n-1}\delta(x_j,y_j)\right].
\end{aligned}$$

So

$$\begin{aligned}
(\nabla^{+}_{p}f(x))^{q}&=\left[\sum_{y\in X^n}\left((f(x)-f(y))^{+}\right)^{p}K^{n}(x,y)\right]^{q/p}\\
&=\left[\sum_{y\in X^n}\left((f(x)-f(y))^{+}\right)^{p}\left\{\frac{1}{n}\Big[(n-1)K^{n-1}(x^{(n-1)},y^{(n-1)})\delta(x_n,y_n)\right.\right.\\
&\qquad\left.\left.+K_n(x_n,y_n)\prod_{j=1}^{n-1}\delta(x_j,y_j)\Big]\right\}\right]^{q/p}.
\end{aligned}$$

Hence

$$
\begin{aligned}
n^{q/p}&\left(\nabla_p^+ f(x)\right)^q = \\
&= \Bigg[(n-1)\sum_{y\in X^n}\left((f(x)-f(y))^+\right)^p K^{n-1}\left(x^{(n-1)},y^{(n-1)}\right)\delta(x_n,y_n) \\
&\qquad + \sum_{y\in X^n}\left((f(x)-f(y))^+\right)^p K_n(x_n,y_n)\prod_{j=1}^{n-1}\delta(x_j,y_j)\Bigg]^{q/p} \\
&= \left[(n-1)\left(\nabla^+_{p,x^{(n-1)}}f(x)\right)^p+\left(\nabla^+_{p,x_n}f(x)\right)^p\right]^{q/p} \\
&\geq \min\left(2^{-1+q/p},1\right)\left[(n-1)^{q/p}\left(\nabla^+_{p,x^{(n-1)}}f(x)\right)^q+\left(\nabla^+_{p,x_n}f(x)\right)^q\right].
\end{aligned}
$$

## 3. A Generalization of Cheeger's inequality

In this section we provide a generalization of Cheeger's inequality; towards this we introduce the constants:

$$k_p^{\pm}(K) = \inf_{f\neq const.}\frac{E_\pi \nabla_p^{\pm} f}{E_\pi (f-m(f))^+},$$

$$k_p(K) = \inf_{f\neq const.}\frac{E_\pi \mid \nabla_p f \mid}{E_\pi |f-m(f)|},$$

where the inf is over all non constant $f: X\to\mathbb{R}$, and where $m(f)$ denotes a median of $f$ (with respect to $\pi$), i.e., $\pi\{f\geq m(f)\}\geq 1/2$, and $\pi\{f\leq m(f)\}\geq 1/2$, and as also well known $m(f)=\arg\min_{a\in\mathbb{R}} E_\pi|f-a|$.

Clearly, $k_p^{\pm}(K)\leq h_p^{\pm}(K)$ and $k_p(K)\leq h_p(K)$. Indeed, let $f=\mathbf{1}_A$. Then,

$$m(\mathbf{1}_A) = \begin{cases} 0 & \text{if } \pi(A)\leq 1/2 \\ 1 & \text{if } \pi(A)\geq 1/2 \end{cases},$$

thus $E_\pi(\mathbf{1}_A - m(\mathbf{1}_A))^+ = \begin{cases} \pi(A) & \text{if } \pi(A)\leq 1/2 \\ 0 & \text{if } \pi(A)\geq 1/2 \end{cases}$, and $E_\pi|\mathbf{1}_A - m(\mathbf{1}_A)| = \min(\pi(A),1-\pi(A))$. In general, $h_1^{\pm}=k_1^{\pm}$ and $h_1=k_1$, and for $1\leq p<+\infty$ these equalities are also true on a two point state space (see the corollary and the remarks following the next theorem). However, for general Markov chains, and except for the case $p=+\infty$ (see [6]), we do not know when equality (or equivalence) between $k_p^{\pm}$ and $h_p^{\pm}$ holds. In fact, the usual ingredient in proving such results, i.e. the coarea (in)equality, fails for the 2-gradient

([15]). It is nevertheless easy to see (from (2.1)) that $k_p^{\pm} \geq C h_p^{\pm}$, whenever $C = \inf_{\{x,y:K(x,y)>0\}} K(x,y)^{1-1/p} < +\infty$, and similarly for $k_p$ and $h_p$.

Let now $N$ be a Young function, that is, $N : \mathbb{R} \to \mathbb{R}$ is convex, even, non-negative, with $N(0) = 0$ and $N(x) > 0$, for all $x \neq 0$. Moreover, assume that

$$C_N = \sup_{x>0} \frac{xN'(x)}{N(x)} < +\infty, \tag{3.1}$$

where $N'$ is a Radon–Nikodym derivative of $N$. Let also $L_N(X,\pi)$ be the Orlicz space of functions $f$ defined on $X$ and such that

$$\|f\|_N = \inf\left\{t > 0 : E_\pi N\left(\frac{f}{t}\right) \leq 1\right\} < +\infty.$$

Finally, and for simplicity, we often write $\|\nabla_p f\|_N$ for $\||\nabla_p f|\|_N$.

**Theorem 3.1.** *Let $k_p^+ = k_p^+(K) > 0$, for $1 \leq p < +\infty$. Then for all functions $f : X \to \mathbb{R}$ such that $m(f) = 0$,*

$$\|f\|_N \leq \frac{C_N}{k_p^+} \|\nabla_p f\|_N \tag{3.2}$$

$$E_\pi N(f) \leq E_\pi N\left(\frac{C_N}{k_p^+} |\nabla_p f|\right). \tag{3.3}$$

**Proof.** We first show that

$$E_\pi \nabla_p^+ N(|f|) \leq E_\pi N'(|f|) \nabla_p^+ |f|. \tag{3.4}$$

Without loss of generality, assume that $f$ is non-negative. Then by the mean value theorem, and since $N'$ is non-decreasing on $(0,+\infty)$, with $N'(0) = 0$,

$$\begin{aligned}
\nabla_p^+ N(f(x)) &= \left(\sum_{y\in X} \left((N(f(x)) - N(f(y)))^+\right)^p K(x,y)\right)^{1/p} \\
&\leq \left(\sum_{y\in X} \left((f(x) - f(y))^+\right)^p |N'(f(x))|^p K(x,y)\right)^{1/p},
\end{aligned}$$

hence

$$\begin{aligned}
E_\pi(\nabla_p^+ N(f)) &\leq \sum_x N'(f(x)) \left(\sum_y \left((f(x) - f(y))^+\right)^p K(x,y)\right)^{1/p} \pi(x) \\
&= E_\pi N'(f)(\nabla_p^+ f).
\end{aligned}$$

This proves (3.4). Now, let $f : X \to \mathbb{R}$ be such that $m(f) = 0$ and $\|f\|_N = 1$, and let $f = f^+ - f^-$. Note that $m(f) = 0$ implies that $m(f^+) = m(f^-) = 0$ and also $m(N(f^+)) = m(N(f^-)) = 0$.

Applying the defining property of $k_p^+$ to $N(f^+)$ and $N(f^-)$, gives

$$\begin{aligned} k_p^+ E_\pi N(f) &= k_p^+ E_\pi N(f^+) + k_p^+ E_\pi N(f^-) \\ &\le E_\pi \nabla_p^+ N(f^+) + E_\pi \nabla_p^+ N(f^-) \\ &\le E_\pi N'(f^+) \nabla_p^+ f^+ + E_\pi N'(f^-) \nabla_p^+ f^- \\ &\le E_\pi N'(|f|) |\nabla_p f|, \end{aligned} \tag{3.5}$$

where the second inequality is just (3.4), while the last one follows from $\nabla_p^+ f^+ \le |\nabla_p f| \mathbf{1}_{f>0}$ and $\nabla_p^+ f^- \le |\nabla_p f| \mathbf{1}_{f<0}$. To finish, we proceed as in the proof of Theorem 3.1 in [5]. Applying Lemma 2.1 of [5] to the functions $|f|$ and $|\nabla_p f\|/\|\nabla_p f\|_N$, we get from (3.5)

$$k_p^+ E_\pi N(f) \le C_N \|\nabla_p f\|_N E_\pi N(f).$$

Hence

$$k_p^+ \le C_N \|\nabla_p f\|_N,$$

and since $\|f\|_N = 1$,

$$\|f\|_N \le \frac{C_N}{k_p^+} \|\nabla_p f\|_N,$$

for all $f$ with $m(f) = 0$. This proves (3.2). To obtain (3.3), we again proceed as in [5] and apply (3.2) to the functions $N_\alpha(x) = N(x)/\alpha$, $\alpha > 0$. ∎

The special case $p = 1$ is interesting for several reasons as explained in the following.

**Corollary 3.2.** *Let $h_1^+ = h_1^+(K) > 0$. Then for all functions $f : X \to \mathbb{R}$ such that $m(f) = 0$,*

$$E_\pi N(f) \le E_\pi N\left(\frac{C_N}{h_1^+} |\nabla_1 f|\right). \tag{3.6}$$

**Proof.** Clearly, $h_p^+ \ge k_p^+ \ge k_1^+$. Thus to prove the corollary, it suffices to establish that $k_1^+ \ge h_1^+$. To show this inequality, we use the following interchange of summation commonly referred to as a co-area (in)equality. (We provide the proof in detail here for completeness; alternatively one can proceed as in the proof of Proposition 3.2 in [4] or the proof of Lemma 1 in [6] for a quicker proof). We do not need to assume that $m(f) = 0$ to prove this interchange; however, for convenience, we assume that $\pi\{f \ge t_M\} \le 1/2$.

The proof can easily be modified otherwise. (See the comments below.) Let $f$ take the values, $-\infty < t_0 < t_1 < \cdots < t_M < +\infty$. i.e. the range of $f$ is $\{t_0, \ldots, t_M\}$. Then

$$\begin{aligned}
E_\pi \nabla_1^+ f &= \sum_x \sum_y (f(x) - f(y))^+ K(x,y)\pi(x) \\
&= \sum_{\substack{x,y \\ f(x)>f(y)}} (f(x) - f(y)) K(x,y)\pi(x) \\
&= \sum_{i=0}^{M-1} \sum_{\substack{x,y \\ f(x)>t_i \\ f(y)\le t_i}} (t_{i+1} - t_i) K(x,y)\pi(x) \\
&= \sum_{i=0}^{M-1} \sum_{x,y} \left(\mathbf{1}_{\{f>t_i\}}(x) - \mathbf{1}_{\{f>t_i\}}(y)\right)^+ (t_{i+1} - t_i) K(x,y)\pi(x) \\
&= \sum_{i=0}^{M-1} E_\pi \left(\nabla_1^+ \mathbf{1}_{\{f>t_i\}}\right) (t_{i+1} - t_i) \\
&\ge \sum_{i=j}^{M-1} h_1^+ \pi \left(\mathbf{1}_{\{f>t_i\}}\right) (t_{i+1} - t_i),
\end{aligned}$$

where $j$ is the largest integer $(<M)$ such that $m(f) \ge t_j$, and the inequality simply uses the definition of $h_1^+$; note that $j \le M-1$, since we assumed $\pi(f \ge t_M) \le 1/2$. (In case, $\pi(f \ge t_M) = 1/2$, we choose $t_{M-1}$ as $m(f)$; and in case $\pi(f \ge t_M) > 1/2$, since then $\pi(f \le t_0) < 1/2$, we can rewrite this proof using the (indicator) functions $\mathbf{1}_{f<t_i}$.) Thus

$$\begin{aligned}
E_\pi \nabla_1^+ f &\ge h_1^+ \sum_{x: f(x)>t_j} \pi(x)(f(x) - t_j) \\
&= h_1^+ \sum_x \pi(x)(f(x) - t_j)^+ \\
&= h_1^+ E_\pi (f - m(f))^+,
\end{aligned}$$

and thus $k_1^+ \ge h_1^+$. ∎

**Remark 3.3.** Since $h_1 = 2h_1^+$, (3.6) becomes

$$E_\pi N(f) \le E_\pi N\left(\frac{2C_N}{h_1} |\nabla_1 f|\right). \tag{3.7}$$

Moreover, in the special case $N(f)=|f|^p$, $1\le p<\infty$, and under reversibility, the extra constant 2 in (3.7) can be omitted (see below for a proof of this claim). However, in general a constant bigger than 1 is needed. Indeed, let $N(x)=2|x|$, for $|x|\le 2$ and $N(x)=x^2$, for $|x|\ge 2$. Now on the state space $\{1,2\}$, let $K(1,1)=K(2,2)=K(1,2)=K(2,1)=1/2$, so that $\pi(1)=\pi(2)=1/2$. Then, the function $f$ defined by $f(1)=3/2$ and $f(2)=4$ satisfies the generalized Cheeger's inequality with a constant greater than unity (but of course, no more than two). To finish this remark, let us prove the claim stated above: Let $N(f)=|f|^p$, $1\le p<\infty$. Then
(a) $E_\pi|\nabla N(|f|)|\le E_\pi N'(|f|)|\nabla_1 f|$,
(b) $\|f\|_N\le \dfrac{C_N}{h_1}\|\nabla_1 f\|_N$.

**Proof.** It suffices to prove (a), since (b) follows from (a) by proceeding as in the end of the proof of Theorem 3.1. Moreover we can also assume that $f$ is non-negative. Let $D=E_\pi N'(|f|)|\nabla_1 f|-E_\pi|\nabla_1 N(|f|)|$. Then, using an incorrect but self-explanatory notation ($p$ is not necessarily an integer),

$$\begin{aligned}
D &= \sum_x p f^{p-1}(x)\sum_y |f(x)-f(y)|K(x,y)\pi(x) - \\
&\qquad - \sum_x\sum_y \left|f^p(x)-f^p(y)\right| K(x,y)\pi(x)\\
&= \sum_x\sum_y \left(\frac{p}{2}f^{p-1}(x)+\frac{p}{2}f^{p-1}(y)\right)|f(x)-f(y)|K(x,y)\pi(x)\\
&\qquad - \sum_x\sum_y |f(x)\\
&\qquad - f(y)|\sum_{j=0}^{p-1}\left(\frac{1}{2}f^j(x)f^{p-1-j}(y)+\frac{1}{2}f^j(y)f^{p-1-j}(x)\right)K(x,y)\pi(x)\\
&\qquad \text{(using reversibility)}\\
&= \sum_x\sum_y |f(x)-f(y)|\sum_{j=0}^{p-1}\frac{1}{2}\left(f^{p-1}(x)-f^j(x)f^{p-1-j}(y)\right)\\
&\qquad + \frac{1}{2}\left(f^{p-1}(y)-f^j(y)f^{p-1-j}(x)\right)K(x,y)\pi(x)\\
&= \frac{1}{2}\sum_x\sum_y |f(x)\\
&\qquad - f(y)|\left\{\sum_{j=0}^{p-1}(f^j(x)-f^j(y))(f^{p-1-j}(x)-f^{p-1-j}(y))\right\}K(x,y)\pi(x)\\
&\ge 0.
\end{aligned}$$

∎

**Remark 3.4.** If $N(x)=|x|^p$, $1\le p<\infty$, then $C_N=p$. In view of Remark 3.3, and under a reversibility assumption, we also get

$$(E_\pi|f-m(f)|^p)^{1/p}\le \frac{p}{h_1}(E_\pi|\nabla_1 f|^p)^{1/p}.$$

Moreover,

$$C(p)(E_\pi|f-E_\pi f|^p)^{1/p}\le (E_\pi|f-m(f)|^p)^{1/p}\le \frac{p}{h_1}(E_\pi|\nabla_1 f|^p)^{1/p},$$

where $C(p)\ge 1/2$ and, of course, $C(2)=1$ and $C(1)=1/2$. Thus,

$$\inf_f \frac{E_\pi|\nabla_1 f|^p}{E_\pi|f-E_\pi f|^p}\ge \left(\frac{C(p)h_1}{p}\right)^p,$$

and in particular for $p=2$, $\inf_f \frac{E_\pi|\nabla_1 f|^2}{E_\pi|f-E_\pi f|^2}\ge \frac{h_1^2}{4}$. Since $E_\pi|\nabla_1 f|^p\le E_\pi|\nabla_p f|^p$, under a reversibility assumption, the "usual" estimate for the spectral gap,

$$\lambda=\inf_f \frac{E_\pi|\nabla_2 f|^2}{2E_\pi|f-E_\pi f|^2}\ge \frac{h_1^2}{8}=\frac{(h_1^+)^2}{2},$$

follows. Analogous to the easy *reverse* inequality, $\lambda\le h_1$, we can also derive a reverse inequality for $N(x)=|x|^p$. Indeed, for any $A\subset X$ $E_\pi|\nabla_p \mathbf{1}_A|^p=E_\pi|\nabla_1\mathbf{1}_A|$. Hence,

$$\inf_f \frac{E_\pi|\nabla_p f|^p}{E_\pi|f-E_\pi f|^p}\le 2^{p-1}h_1. \tag{3.8}$$

This last inequality also admits a version for $\nabla_p^+$ and $h_1^+$ as well as for Orlicz norms where the constant depends on $N$.

**Remark 3.5.** The preceeding remark brings us to a key question. How does $h^2, h_2^+, h_2^-$ compare to $\lambda$? We know (say, under a reversibility assumption) that

$$\frac{h_1^2}{8}=\frac{(h_1^\pm)^2}{2}\le\lambda\le h_1=2h_1^\pm\le 2\min(h_2^+,h_2^-)$$

and it is easy to show that,

$$\left(h_2^+\right)^2\le h_1^+ \text{ and } h_2^\pm\le\left(\sup_{x,y:K(x,y)>0}\frac{1}{\sqrt{K(x,y)}}\right)h_1^\pm.$$

Moreover, each of the above inequality can be tight as the examples presented below show. In turn, these inequalities imply that

$$\max\left(\left(h_2^+\right)^2, \left(h_2^-\right)^2\right) \leq 2\left(\sup_{x,y:K(x,y)>0} \frac{1}{K(x,y)}\right)\lambda.$$

However, for the plus version, we do not know wether or not the factor involving $K(x,y)$ on the above right hand side can be replaced by a positive absolute constant. We offer it as an open question/conjecture to the reader (once more, see also Concluding Remarks). Note that neither $\lambda \geq C h_2^2$, nor $\lambda \geq C(h_2^-)^2$, where $C$ are absolute constants, hold true. Indeed, for the 2-state Markov chain on $\{0,1\}$ with $K(0,0) = K(1,0) = 1-r$ and $K(1,1) = K(0,1) = r$, and $r < 1/2$. The invariant distribution is $\pi(1) = 1 - \pi(0) = r$. Then, $h_2 = \sqrt{1-r} + \dfrac{1-r}{\sqrt{r}}$, $h_2^- = \dfrac{1-r}{\sqrt{r}}$, $h_2^+ = \sqrt{1-r}$ and $\lambda = 1$. Thus, $\lambda$ is independent of $r$, but $h_2, h_2^- \to \infty$ as $r \to 0$.

**Some examples.** In each of the following examples, the simple random walk is our (reversible) Markov chain on the corresponding graph and $\pi$ is the uniform probability measure. (Recall that for $a > 0$ and $b > 0$, the notation $a = \Theta(b)$ indicates that the ratio of the two quantities is bounded below and above by absolute constants.)

**1** *A $d$-ary tree.* The simple random walk on the $d$-ary tree of height $H$ has $h_1^+ = \Theta(1/d^{H-1})$, $h_2^+ = \Theta\left(\sqrt{d}/(d^{H-1})\right)$, and $\lambda = \Theta(1/d^{H-1})$. See [19] for the estimate on $\lambda$, whereas the set of vertices in the left subtree of the root vertex forms an extremal set for $h_1^+$ and $h_2^+$.

**2** *The discrete cube.* For the hypercube on $2^n$ vertices, $\{0,1\}^n$, it is well known (e.g., [1], [2]) that $\lambda = 1/n$, $h_1^+ = 1/n$ and easy to verify that $h_2^+ = \Theta(1/\sqrt{n})$ (see Remark 4.5).

**3** *The $n$-cycle.* For the cycle of length $n$, it is classical that $\lambda = \Theta(1/n^2)$ and easily verified that $h_1^+ = \Theta(1/n) = h_2^+$.

**4** *A necklace.* Consider the graph $N_{n,d}$ obtained by replacing each vertex of a cycle of length $n$ by a complete graph $K_d$ on $d$ vertices as shown in Figure 1. It is straightforward to check that the simple random walk on this graph has $h_1^+ = \Theta(1/(nd^2))$, and $h_2^+ = \Theta\left(1/(nd^{3/2})\right)$. Using the techniques in [12], it is not hard to check that $\lambda = \Theta(1/(n^2 d^2))$. This is an example in which the inequality $\lambda \geq C(h_2^+)^2$, if true, would provide a better estimate on the spectral gap than the usual Cheeger type inequality.

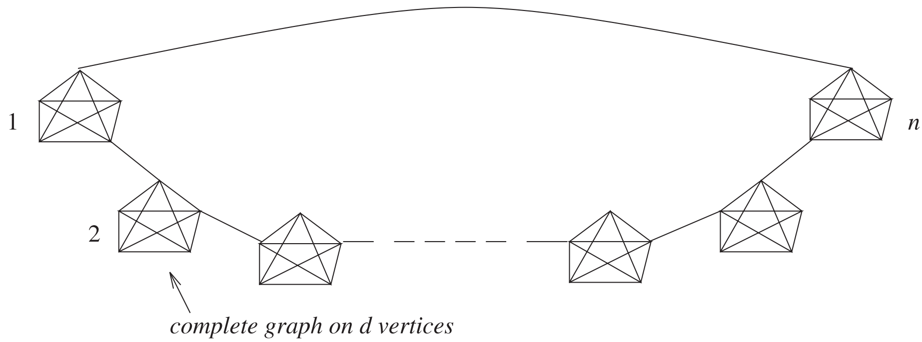


**Fig. 1.** Necklace graph $N_{n,d}$

## 4. Induction and main result

With the framework of the previous sections, we now state

**Theorem 4.1.** *For any* $1 \le p \le 2$,

$$\frac{1}{n^{1/p}} \min_{1\le i\le n} h_p(K_i) \ge h_p(K^n) \ge \frac{1}{2\sqrt{6}n^{1/p}} \min_{1\le i\le n} k_p^+(K_i). \tag{4.1}$$

*In particular, let* $\delta_i = \inf_{x,y:K_i(x,y)>0} K_i(x,y) > 0$, *for all* $1 \le i \le n$, *and let the chains be reversible, then*

$$h_p(K^n) \ge \frac{1}{2\sqrt{6}n^{1/p}} \min_{1\le i\le n} h_1^+(K_i) \ge \frac{1}{4\sqrt{6}n^{1/p}} \min_{1\le i\le n} \left(\delta_i^{(p-1)/p} h_p(K_i)\right),$$

$1 \le p \le 2$.

The following lemma contains a key induction and is essential to the proof.

**Lemma 4.2.** *Let* $1 \le p \le 2$ *and let* $D_p$ *denote any one of* $\nabla_p^{\pm}$ *or* $|\nabla_p|$. *Let* $C > 0$ *be such that for each* $X_i$ $(i = 1, \dots, n)$,

$$\sum_{x\in X_i} \sqrt{1 + f^2(x)}\pi_i(x) \le \sum_{x\in X_i} \sqrt{1 + C^2(D_p f(x))^2}\pi_i(x) \tag{4.2}$$

*for all functions* $f$ *on* $X_i$ *with* $m(f) = 0$. *Then*

$$E_{\pi^n}(D_p f) \ge \frac{2}{\sqrt{6}n^{1/p}C} \mathrm{Var}_{\pi^n}(f), \quad \textit{for all } f : X^n \to \mathbb{R}. \tag{4.3}$$

**Proof.** We only prove the lemma for $\nabla_p^+$, the other cases being similar. Let $C>0$ be such that (4.2) is satisfied. Note that if for all $x\in X_i$, $0\le f(x)\le a$, then $|f(x)-m(f)|\le a$ and $\sqrt{1+(f-m(f))^2}\ge 1+\ell(a)(f-m(f))^2$, where $\ell(a)=\left(\sqrt{1+a^2}-1\right)/a^2$ is the optimal constant $\ell$ such that $\sqrt{1+t^2}\ge 1+\ell t^2$, for all $|t|\le a$. Therefore,

$$\begin{aligned}\sum_{x\in X_i}(1+(f(x)-m(f))^2)^{1/2}\pi_i(x) &\ge 1+\ell(a)\sum_{x\in X_i}(f(x)-m(f))^2\pi_i(x)\\ &\ge 1+\ell(a)\mathrm{Var}_{\pi_i}(f).\end{aligned}$$

Thus from (4.2),

$$1+\ell(a)\mathrm{Var}_{\pi_i}(f)\le\sum_{x\in X_i}\sqrt{1+C^2(\nabla_p^+f(x))^2}\pi_i(x), \tag{4.4}$$

for all functions $f$ on $X_i$ with $0\le f\le a$, and for $i=1,\dots,n$. By induction on $n$, we are going to extend (4.4) to all functions $f:X^n\to[0,a]$ and prove that

$$1+L(a)\mathrm{Var}_{\pi^n}(f)\le\sum_{x\in X^n}\sqrt{1+C^2n^{2/p}(\nabla_p^+f(x))^2}\pi^n(x), \tag{4.5}$$

where $L=L(a)$ is an arbitrary positive function such that

$$L(1+La^2/4)\le\ell\left(\frac{a}{(1+La^2/4)}\right),\qquad L(a)\le\ell(a). \tag{4.6}$$

The proof of the inductive step presented below is very similar to the proof of Lemma 4.1 in [5]. Assuming (4.5) is proved, use $\sqrt{1+t^2}\le 1+t$, $t\ge 0$, to get

$$L(a)\mathrm{Var}_{\pi^n}(f)\le Cn^{1/p}\sum_{x\in X^n}\nabla_p^+f(x)\pi^n(x),$$

for any $f:X^n\to[0,a]$. Now, proceeding as in the end of the proof of Lemma 4.2 in [5] gives $L(a)\ge 2/\sqrt{6}$. (Note that $L(a)$ above is $aL(a)$ in [5].)

**Proof of (4.5): the inductive step.** To prove this induction step, take a function $f:X^{n+1}\to[0,a]$ and introduce the function

$$\alpha(x_{n+1})=\sum_{x\in X^n}f(x,x_{n+1})\pi^n(x),\qquad\text{for } x_{n+1}\in X_{n+1}.$$

Note that $\alpha : X_{n+1} \to [0,a]$, since $\pi^n$ is a measure on $X^n$. We have

$$\begin{aligned}
\nabla_p^+ \alpha(x_{n+1}) &= \left( \sum_{y \in X_{n+1}} ((\alpha(x_{n+1}) - \alpha(y))^+)^p \, K_{n+1}(x_{n+1}, y) \right)^{1/p} \\
&= \left( \sum_{y \in X_{n+1}} \left( \Big( \sum_{x \in X^n} (f(x, x_{n+1}) - f(x,y)) \pi^n(x) \Big)^+ \right)^p K_{n+1}(x_{n+1}, y) \right)^{1/p} \\
&\le \sum_{x \in X^n} \left( \sum_{y \in X_{n+1}} ((f(x, x_{n+1}) - f(x,y))^+)^p \, K_{n+1}(x_{n+1}, y) \right)^{1/p} \pi^n(x) \\
&= \sum_{x \in X^n} (\nabla_{p, x_{n+1}}^+ f(x, x_{n+1})) \pi^n(x),
\end{aligned}$$

using the “integral” Minkowski inequality. Thus

$$\nabla_p^+ \alpha(x_{n+1}) \le \sum_{x \in X^n} (\nabla_{p, x_{n+1}}^+ f(x, x_{n+1})) \pi^n(x). \tag{4.7}$$

where $\nabla_{p,x_{n+1}}^+ f$ is the gradient with respect to the coordinate $x_{n+1}$. By Lemma 2.1, we have, for $1 \le p \le 2$,

$$(n+1)^{2/p} (\nabla_p^+ f)^2 \ge \left[ n^{2/p} (\nabla_{p,x}^+ f)^2 + (\nabla_{p,y}^+ f)^2 \right], \quad \text{for } x \in X^n, \quad y \in X_{n+1}.$$

Thus

$$\begin{aligned}
&\sum_{x \in X^n} (1 + C^2 (n+1)^{2/p} (\nabla_p^+ f)^2)^{1/2} \pi^n(x) \\
&\qquad \ge \sum_{x \in X^n} (1 + C^2 n^{2/p} (\nabla_{p,x}^+ f)^2 + C^2 (\nabla_{p,y}^+ f)^2)^{1/2} \pi^n(x).
\end{aligned}$$

Use the elementary inequality

$$\sum \sqrt{u^2 + v^2} \ge \sqrt{\left(\sum u\right)^2 + \left(\sum v\right)^2},$$

with $u = \sqrt{1 + C^2 n^{2/p} (\nabla_{p,x}^+ f)^2}$ and $v = C(\nabla_{p,y}^+ f)$, so that

$$\begin{aligned}
&\sum_{x \in X^n} (1 + C^2 (n+1)^{2/p} (\nabla_p^+ f)^2)^{1/2} \pi^n(x) \\
&\ge \sqrt{ \left[ \sum_{x \in X^n} (1 + C^2 n^{2/p} (\nabla_{p,x}^+ f)^2)^{1/2} \pi^n(x) \right]^2 + C^2 \left( \sum_{x \in X^n} \nabla_{p,y}^+ (f) \pi^n(x) \right)^2 } \\
&\ge \sqrt{ (1 + L(a) \mathrm{Var}_{\pi^n}(f))^2 + C^2 (\nabla_p^+ \alpha(y))^2 },
\end{aligned}$$

where the last inequality follows from the induction hypothesis as well as (4.7). Since $\mathrm{Var}_{\pi^n}(f) \le a^2/4$, and since $\sqrt{1+t^2}-t$, $t>0$, is decreasing in $t$ we get

$$
\begin{aligned}
&\sum_{x\in X^n} \sqrt{1+C^2(n+1)^{2/p}(\nabla_p^+ f)^2}\pi^n(x) - (1+L(a)\ \mathrm{Var}_{\pi^n}(f)) \\
&\qquad \ge \sqrt{(1+L(a)\ \mathrm{Var}_{\pi^n}(f))^2 + C^2(\nabla_p^+ \alpha(y))^2} - (1+L(a)\ \mathrm{Var}_{\pi^n}(f)) \\
&\qquad \ge \sqrt{(1+L(a)a^2/4)^2 + C^2(\nabla_p^+ \alpha(y))^2} - (1+L(a)a^2/4) \\
&\qquad \ge (1+L(a)a^2/4)\sqrt{1+C^2(\nabla_p^+ \alpha_1(y))^2} - (1+L(a)a^2/4), \qquad (4.8)
\end{aligned}
$$

where $\alpha_1 = \dfrac{\alpha}{(1+L(a)a^2/4)}$.

Summing (4.8) over $y\in X_{n+1}$ gives

$$
\begin{aligned}
&\sum_{(x,y)\in X^{n+1}} (1+C^2(n+1)^{2/p}(\nabla_p^+ f)^2)^{1/2}\pi^{n+1}(x,y) \\
&\qquad\qquad - \sum_{y\in X_{n+1}} (1+L(a)\ \mathrm{Var}_{\pi^n}(f))\pi(y) \\
&\quad \ge (1+L(a)a^2/4) \sum_{y\in X_{n+1}} \sqrt{1+C^2(\nabla_p^+ \alpha_1(y))^2}\pi(y) - (1+L(a)a^2/4).
\end{aligned}
$$

Using (4.4) with $i=n+1$, we get that the above is

$$
\begin{aligned}
&\ge (1+L(a)a^2/4)(1+\ell(a_1)\mathrm{Var}_{\pi_{n+1}}(\alpha_1)) - (1+L(a)a^2/4) \\
&= \frac{1}{(1+L(a)a^2/4)}[\ell(a_1)\mathrm{Var}_{\pi_{n+1}}(\alpha)], \quad \text{where} \quad a_1 = \frac{a}{(1+L(a)a^2/4)}.
\end{aligned}
$$

That is,

$$
\begin{aligned}
&\sum_{(x,y)\in X^{n+1}} \sqrt{1+C^2(n+1)^{2/p}(\nabla_p^+ f)^2}\pi^{n+1}(x,y) \\
&\quad \ge 1 + L(a) \sum_{y\in X_{n+1}} \mathrm{Var}_{\pi^n}(f)\pi_{n+1}(y) + \frac{\ell(a_1)\mathrm{Var}_{\pi_{n+1}}(\alpha)}{(1+L(a)a^2/4)}
\end{aligned}
$$

and we wish to show that this last expression is bounded below by

$$
1 + L(a)\mathrm{Var}_{\pi^{n+1}}(f).
$$

Putting $\beta(y)=\sum_{x\in X^n} f^2(x,y)\pi^n(x)$ we have

$$\mathrm{Var}_{\pi^n}(f) = \sum_{x\in X^n} f^2(x,y)\pi^n(x) - \left(\sum_{x\in X^n} f(x,y)\pi^n(x)\right)^2 = \beta(y) - (\alpha(y))^2.$$

Also it is easy to verify that

$$\mathrm{Var}_{\pi^{n+1}}(f) = \sum_{y\in X_{n+1}} \beta(y)\pi_{n+1}(y) - \left(\sum_{y\in X_{n+1}} \alpha(y)\pi_{n+1}(y)\right)^2.$$

We now wish to show that

$$\begin{aligned} L(a)&\left(\sum_{y\in X_{n+1}} \beta(y)\pi_{n+1}(y) - \sum_{y\in X_{n+1}} \alpha^2(y)\pi_{n+1}(y)\right) \\ &\quad + \frac{\ell(a_1)}{(1+L(a)a^2/4)}\,\mathrm{Var}_{\pi_{n+1}}(\alpha) \\ &\geq L(a)\left[\sum_{y\in X_{n+1}} \beta(y)\pi_{n+1}(y) - \left(\sum_{y\in X_{n+1}} \alpha(y)\pi_{n+1}(y)\right)^2\right]. \end{aligned}$$

That is,

$$\frac{\ell(a_1)}{(1+L(a)a^2/4)}\,\mathrm{Var}_{\pi_{n+1}}(\alpha) \geq L(a)\mathrm{Var}_{\pi_{n+1}}(\alpha),$$

i.e.,

$$L(a)(1+L(a)a^2/4) \leq \ell(a_1),$$

where

$$a_1 = \frac{a}{(1+L(a)a^2/4)}.$$

We want

$$L(a)(1+L(a)a^2/4) \leq \ell\left(\frac{a}{(1+L(a)a^2/4)}\right),$$

and $L(a)\leq\ell(a)$ for the induction to work. Finally, note that in Lemma 4.2 of [5], the above conditions *are* shown to hold with the choice $L(a)\geq 2/\sqrt{6}$, thus establishing, for $n\geq 1$,

$$1 + L(a)\mathrm{Var}_{\pi^n}(f) \leq \sum_{x\in X^n} \sqrt{1+C^2n^2(\nabla_p^+ f(x))^2}\pi^n(x). \qquad \blacksquare$$

We are now ready to prove our main result.

**Proof of Theorem 4.1.** For the Young function $N(x)=\sqrt{1+x^2}-1$, we have $C_N=2$. Now, combine Theorem 3.1 and Lemma 4.2. By (3.3), the inequality (4.2) holds with $C=2/\min_{1\le i\le n} k_p^+(K_i)$, hence from (4.3) for $D_p=|\nabla_p|$,

$$E_{\pi^n}|\nabla_p f| \ge \frac{\min_{1\le i\le n} k_p^+(K_i)}{\sqrt{6}n^{1/p}}\mathrm{Var}_{\pi^n}(f),$$

for all $f$ with $m(f)=0$. Now, let $f=\mathbf{1}_A$ for $A\subseteq X^n$ with $\pi^n(A)\le 1/2$, so that $m(f)=0$. Then

$$\sum_{x\in X^n} |\nabla_p f(x)|\pi^n(x) = \sum_{x\in X^n}\left(\sum_{y\in X^n} |\mathbf{1}_A(x)-\mathbf{1}_A(y)|\, K^n(x,y)\right)^{1/p}\pi^n(x);$$

and

$$\mathrm{Var}_{\pi^n}(\mathbf{1}_A) = \pi^n(A)(1-\pi^n(A)) = \pi^n(A)\pi^n(A^c).$$

Thus we have

$$\begin{aligned} E_{\pi^n}|\nabla_p \mathbf{1}_A| &\ge \frac{\min_{1\le i\le n} k_p^+(K_i)}{\sqrt{6}n^{1/p}}\pi^n(A)(1-\pi^n(A)) \\ &\ge \frac{\min_{1\le i\le n} k_p^+(K_i)}{2\sqrt{6}n^{1/p}}\min(\pi^n(A),1-\pi^n(A)), \end{aligned} \tag{4.9}$$

and the inequality (4.1) follows. To prove the last assertion of the theorem, recall that $k_p^+\ge k_1^+=h_1^+=h_1/2$, and use (2.1) for $|\nabla_p|$. (Again, as previously mentioned, reversibility is not needed here). ∎

**Remark 4.3.** A careful analysis of the proof of Theorem 4.1 actually shows that for any $1\le p\le 2$,

$$\begin{aligned} \frac{1}{n^{1/p}}\min_{1\le i\le n} h_p^+(K_i) \ge h_p^+(K^n) &\ge \frac{1}{2\sqrt{6}n^{1/p}}\min_{1\le i\le n} k_p^+(K_i) \\ &\ge \frac{1}{2\sqrt{6}n^{1/p}}\min_{1\le i\le n} h_1^+(K_i). \end{aligned} \tag{4.10}$$

Indeed, the first inequality is clear, while the second inequality is obtained after the following modifications to the present proof: in the definition of $k_p^+(K)$ take, without loss of generality, the infimum over non-negative non-constant functions. Then, in Theorem 3.1 and for any $f\ge 0$, note that (3.1) and (3.2) continue to hold with $|\nabla_p f|$ replaced by $\nabla_p^+ f$. The rest of the proof is then changed accordingly with these new estimates.

It is natural to wonder whether or not (4.10) could be sharpened by replacing $k_p^+$ by $h_p^+$. While we do not know if such an inequality is true in general, we observe that it is true on the weighted hypercube. Indeed, let $X=\{0,1\}$, and let $K(\cdot,\cdot)$ be defined arbitrarily. Then, $h_p^+(K)=k_p^+(K)=\max(K(0,1),K(1,0))^{1/p}$, for $1\le p<\infty$ and $h_\infty^+=k_\infty^+=1$. By the same arguments, the unweighted hypercube, where $K(0,1)=K(1,0)=1/2$, shows that (1.4) and (4.10) are tight since then $h_2(K^n)=\Theta(h_1(K)/\sqrt{n})$ and $h_2^+(K^n)=\Theta\left(h_1^+(K)/\sqrt{n}\right)$. Also note that it follows from (4.10) that $h_p^+(K^n)\ge 1/(2\sqrt{6}n^{1/p})\min_{1\le i\le n}\left(\delta_i^{(p-1)/p}h_p^+(K_i)\right)$ where $\delta_i$ is as given in Theorem 4.1. Hence if $\delta_i$ is uniformly bounded from below (e.g. as in simple random walks on bounded graphs), then such a sharpening certainly holds true.

**Remark 4.4.** The inequalities, $h_2(K^n)\ge C(h_2(K)/\sqrt{n})$ and $h_2^-(K^n)\ge C(h_2^-(K)/\sqrt{n})$, for some absolute constants $C$ do not hold as the example of the weighted hypercube shows. As above, let $X=\{0,1\}^n$, be endowed with the product measure $\pi^n(x)=r^m(1-r)^{n-m}$, for $x\in X$, where $m=\sum_{i=1}^n x_i$, and $0<r<1$, and let also, $K(1,1)=K(0,1)=r$, and $K(0,0)=K(1,0)=1-r$. As observed in Remark 3.5, $h_2(K)=\sqrt{1-r}+\dfrac{1-r}{\sqrt{r}}$, and $h_2^-(K)=\dfrac{1-r}{\sqrt{r}}$. To bound $h_2(K^n)$ and $h_2^-(K^n)$ from above, take $A=\{(0,0,\dots,0)\}$. Then $E_{\pi^n}|\nabla_2\mathbf{1}_A|=\sqrt{K(0,1)}(1-r)^n+\sqrt{K(1,0)n}r(1-r)^{n-1}$ and $E_{\pi^n}\nabla_2^-\mathbf{1}_A=\sqrt{K(1,0)n}r(1-r)^{n-1}$, and $\pi^n(A)=(1-r)^n$. Thus $h_2(K^n)\le\sqrt{K(0,1)}+r\sqrt{K(1,0)n}/(1-r)=\sqrt{r}+r\sqrt{n/(1-r)}$ and similarly $h_2^-(K^n)\le r\sqrt{n/(1-r)}$. Now choosing $r=1/n$ (and since for this choice $\pi^n(A)\le 1/2$), we see that $\lim_{n\to+\infty}h_2(K^n)=0$ and $\lim_{n\to+\infty}h_2^-(K^n)=0$. On the other hand, $h_2(K)/\sqrt{n}$ and $h_2^-(K)/\sqrt{n}$ being bounded below by $(1-r)/\sqrt{rn}$ remain bounded away from zero when $rn=1$.

**Remark 4.5.** Theorem 4.1 does not hold for $p>2$, and this is to be expected in view of the tensorization property of the various spectral gaps associated to the various gradients (see [17]). As shown next, the isoperimetry of the *Hamming balls of a fixed radius* in the (unweighted) hypercube provides a counterexample. Using the notation of the previous remarks with $r=1-r=1/2$, let $A\subset\{0,1\}^n$ be given by:

$$A=\{(x_1,\dots,x_n)\in\{0,1\}^n:\sum_i x_i\le\lfloor n/2\rfloor\}.$$

(Assume that $n$ is odd, for convenience, so that $\pi^n(A)=1/2$.) For $2\le p<\infty$,

$$E_\pi \nabla_p^+ \mathbf{1}_A = \binom{n}{\lfloor n/2 \rfloor} \left(\lceil n/2 \rceil (1/2n)\right)^{1/p},$$

and hence

$$\frac{E_\pi \nabla_p^+ \mathbf{1}_A}{\pi(A)} = (1+o(1)) \frac{1}{\sqrt{\pi n}} (2)^{3/2-2/p},$$

(and similarly $\dfrac{E_\pi |\nabla_p \mathbf{1}_A|}{\pi(A)} = 2(1+o(1)) \dfrac{1}{\sqrt{\pi n}} (2)^{3/2-2/p}$). Thus, for $2\le p\le\infty$,

$$h_p^+(K^n) \le (1+o(1)) \frac{1}{\sqrt{\pi n}} (2)^{3/2-2/p}.$$

Hence a lower bound result with the $1/n^{1/p}$ scaling does not hold for $p>2$. In view of [17], one would expect a scaling in $1/\sqrt{n}$, for any $p>2$. The present proof of Theorem 4.1 can be modified to give such a statement. However, the reader will find in [24] a simpler proof of a lower bound theorem, scaling in $1/\sqrt{n}$, for $p>2$.

## 5. Graph products and isoperimetric numbers

In this section we very briefly reinterpret our results in the setting of Cartesian product of graphs.

Given two graphs $G_1$ and $G_2$, there are at least three notions of products of $G_1$ and $G_2$; namely, the strong product $G_1 \boxtimes G_2$, the Cartesian product $G_1 \square G_2$, and the weak product $G_1 \times G_2$. Given $G_1=(V_1,\mathcal{E}_1)$ and $G_2=(V_2,\mathcal{E}_2)$, $G_1 \square G_2 = (V,\mathcal{E})$ is defined as follows: $V=V_1\times V_2$ is the Cartesian product of the sets $V_1$ and $V_2$, and

$$\mathcal{E} = \Big\{ \{\{v_1,v_2,\},\{v_1',v_2'\}\} : v_1 = v_1' \text{ and } \{v_2,v_2'\}\in\mathcal{E}_2 \\ \text{or } \{v_1,v_1'\}\in\mathcal{E}_1 \text{ and } v_2 = v_2' \Big\}.$$

In general, the Cartesian product is considered to be the most interesting one (see [21] for various properties of these products).

In our isoperimetric framework, this is also justified for the following reason: If $G$ is an *undirected* graph, its (edge) *isoperimetric number*, $i(G)$, defined via

$$i(G) = \min_{\emptyset\ne A\not\subseteq V} \frac{\#\mathcal{E}(A,A^c)}{\min(\#A,\#A^c)},$$

where $\mathcal{E}(A, A^c)$ is the set of edges with one end in $A$ and the other end in $A^c$, has received extensive attention ([2], [23]).

Now, the definitions of the three products are such that the set of vertices in each case is identical (namely, the Cartesian product of the individual vertex sets), but the sets of edges satisfy the following property:

$$\mathcal{E}(G_1 \square G_2) \cap \mathcal{E}(G_1 \times G_2) = \emptyset \text{ and } \mathcal{E}(G_1 \square G_2) \cup \mathcal{E}(G_1 \times G_2) = \mathcal{E}(G_1 \boxtimes G_2).$$

In short, $\boxtimes = \square \cup \times$. This implies that $i(G_1 \boxtimes G_2) \geq i(G_1 \square G_2) + i(G_1 \times G_2)$, since the isoperimetric number of an *edge disjoint union* of graphs is at least the sum of the isoperimetric numbers of the individual graphs. Thus lower bounds for the isoperimetric number of the strong product can be obtained via lower bounds for the isoperimetric number of the Cartesian product. Note that the above inequality is *not* true, if we allow the presence of *self-loops* in our graphs. For example, if there is a self-loop on every vertex of $G_1$ and $G_2$, then $G_1 \boxtimes G_2 = G_1 \times G_2$, and so obviously, $i(G_1 \boxtimes G_2) = i(G_1 \times G_2)$.

Next, the following example (which involves no self-loops) shows that it is, in general, not possible to prove both lower and upper dimension-free bounds for the strong product. Consider the strong product (as defined above) of $n$ copies of a single edge; i.e. $G^n = K_{2^n} = K_2 \boxtimes K_2 \boxtimes \cdots \boxtimes K_2$, where $K_m$ is the complete graph on $m$ vertices. While $i(K_2) = 1$, we have $i(K_{2^n}) = 2^{n-1}$. (Note here that the isoperimetric numbers of the Cartesian and the weak product of $n$ copies of $K_2$ are 1 and 0, respectively.) It is also easy to produce examples $G_1$, $G_2$ such that $i(G_1 \times G_2) = 0$, while $i(G_1), i(G_2) > 0$. (For example, take $G_1$ and $G_2$ to be paths of length $n$; then $G_1 \times G_2$ is a disconnected graph.)

Let $G = (V, \mathcal{E})$ be a finite connected and, say, undirected graph, and let $A_G = A(G)$ denote the *adjacency matrix* of $G$; that is, $A_G(x, y) = 1$ if $\{x, y\} \in \mathcal{E}$ and 0 otherwise. (Henceforth, we shall assume all graphs are finite, and connected.) Further, let $\pi$ be an arbitrary probability measure on the set of vertices $V$. For $1 \leq p < \infty$, let

$$i_p^+(G) = i_p^+(G, \pi) = \min_{\substack{A \subset V \\ 0 < \pi(A) \leq 1/2}} \frac{\sum_{x \in A} \left( \sum_{y \in A^c} A_G(x, y) \right)^{1/p} \pi(x)}{\pi(A)},$$

(The case $p = +\infty$ is studied in detail in [6], [24] and so not treated here. Note however, that for $p = +\infty$, the graph and the Markov chain gradients coincide).

Given our definitions of $k_p^+$ and $h_p^+$ in the context of Markov kernels, one arrives at the above definition by first defining the "discrete gradient" in the graph setting as follows: For any graph $G$, the role of $K$ is played by the adjacency matrix $A_G$. Thus given any function $f : V \to \mathbb{R}$, for $x \in V$, one sets

$$\begin{aligned}\nabla_p^+ f(x) &= \left(\sum_y ((f(x)-f(y))^+)^p A_G(x,y)\right)^{1/p}\\ &= \left(\sum_{y\sim x} ((f(x)-f(y))^+)^p\right)^{1/p},\end{aligned}$$

where $y \sim x$ denotes the fact that $y$ is adjacent to $x$. Thus (with more abuse of notation) one can also define

$$j_p^+(G) = \inf_{f\neq 0} \frac{E_\pi \nabla_p^+ f}{E_\pi (f - m(f))^+},$$

where as before $m(f)$ is a median of $f$ with respect to $\pi$. Of course, mimicking our previous definitions, leads us to $\nabla_p^+ f$ and $|\nabla_p f|$. Hence, for $1 \le p < \infty$, one has

$$i_p(G) = \min_{0<\pi(A)\le\frac{1}{2}} \frac{\sum_{x\in A}\left(\sum_{y\in A^c} A_G(x,y)\right)^{1/p}\pi(x) + \sum_{x\in A^c}\left(\sum_{y\in A} A_G(x,y)\right)^{1/p}\pi(x)}{\pi(A)},$$

and similarly one defines $j_p$, $i_p^-$ and $j_p^-$. Let us finally note here that in contrast to the Markov case, the graph gradients are *non-increasing* in $p$.

A natural choice for the probability measure $\pi$ is the uniform measure over $V$. It is then easy to see that if $\pi(x) = 1/\#V$, for all $x \in V$, then $i_1^+(G) = i_1^-(G) = i_1(G)/2$ is just the usual isoperimetric number of the graph $G$. Hence, a co-area-based proof as in the Markov chain case also shows that $i_1^\pm(G) = j_1^\pm(G)$, and $i_1(G) = j_1(G)$.

We now consider the Cartesian product of $n$ graphs, $G_1 = (V_1, \mathcal{E}_1), \dots, G_n = (V_n, \mathcal{E}_n)$. For brevity of notation, we replace $G_1 \square \cdots \square G_n$ by $G^n$. Note that $G^n = (V^n, \mathcal{E}^n)$, where $V^n = V_1 \times V_2 \times \cdots \times V_n$, and $\mathcal{E}^n$ is defined inductively using the rule previously defined for the (Cartesian) product of two graphs. Note that trivially

$$i(G^n) \le \min_{1\le m\le n} (i(G_m)),$$

and that equality can hold as in the case of product of edges (see [14]) or more generally product of paths (see [23]). It is also easy to produce examples of $G_1, G_2$ such that

$$i(G_1 \square G_2) < \min(i(G_1), i(G_2)).$$

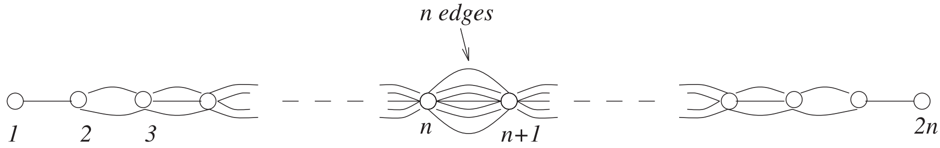


**Fig. 2.**

(See [23], for example.)

In the graph setting, the dependency in $n$ produced by the product structure is different from the one given by Lemma 2.1. In fact, with corresponding notions of "partial gradients", and a similar proof, we have:

**Lemma 5.1.** *For any* $1 \le p \le +\infty$ *and* $x \in X^n$,

$$(D_p f(x))^p = \left(D_{p,x^{(n-1)}} f(x)\right)^p + (D_{p,x_n} f(x))^p ,$$

*where* $D_p$ *is anyone of* $\nabla_p^{\pm}$ *or* $|\nabla_p|$. *Hence,*

$$(D_p f(x))^p = \sum_{i=1}^{n} (D_{p,x_i} f(x))^p .$$

With this lemma, the results in Section 2 and Section 3 carry over to our new setting and so does the proof of Theorem 4.1. So, recalling Remark 4.3 we have:

**Theorem 5.2.** *For* $(G^n, \pi^n)$ *as defined above, and any* $1 \le p \le 2$,

$$\min_{1 \le m \le n} i_p^+(G_m) \ge i_p^+(G^n) \ge \frac{1}{2\sqrt{6}} \min_{1 \le m \le n} j_p^+(G_m). \tag{5.1}$$

**Remark 5.3.** In analogy with the question raised at the end of Section 3, we wonder if there is an absolute constant $C > 0$ such that the spectral gap of any graph $G$ is lower bounded by $C(i_2^+(G))^2$. Note that for the graph $G$ shown in Figure 2, the maximum degree is $\Delta(G) = 2n - 1$. Then, it is easy to check that $i(G) = 1$, $i_2^+ = \Theta(1/\sqrt{n})$, and that the spectral gap is $\Theta(1/n)$.

## 6. Concluding Remarks

- **A direct elementary proof of the $\ell^1$-product theorem.**

A first version of the present paper [16] contained the theorem stated in the introduction for $p = 1$. Having known our results, Fan Chung and subsequently Jean Pierre Tillich noted a simple direct proof of the following strengthening (of the constant) in our result (1.3), for $p=1$.

$$h_1(K^n) \geq \frac{1}{2n} \min_{1\leq i\leq n} h_1(K_i). \tag{6.1}$$

Below, we give an even simpler proof of a similar result. As defined below, $\tilde{h}_1(K)$ is within a factor of 2 of $h_1(K)$, and crucially, $\tilde{h}_1(K)$ satisfies an appropriate product inequality with *equality*. More precisely, let

$$\tilde{h}_1(K) = \min_{0<\pi(A)<1} \frac{E_\pi|\nabla_1 \mathbf{1}_A|}{\pi(A)\pi(A^c)}.$$

Then, note that $\mathrm{Var}_\pi(\mathbf{1}_A)=\pi(A)\pi(A^c)$, while the denominator in the definition of $h_1$ is

$$\min(\pi(A),\pi(A^c)) = E_\pi\left|\mathbf{1}_A - m(\mathbf{1}_A)\right|,$$

where, as before, $m(\mathbf{1}_A)$ is a median of $\mathbf{1}_A$ with respect to $\pi$.

Clearly, $\tilde{h}_1(K)/2 \leq h_1(K) \leq \tilde{h}_1(K)$. Moreover, we claim the following, which in turn implies (6.1).

**Proposition 6.1.** $\tilde{h}_1(K^n)=\dfrac{1}{n}\min\limits_{1\leq i\leq n} \tilde{h}_1(K_i)$.

**Proof.** It is straightforward to show that

$$\tilde{h}_1(K^n) \leq \frac{1}{n} \min_{1\leq i\leq n} \tilde{h}_1(K_i).$$

For the other direction it follows from Lemma 2.1, with $p=q=1$, that for all $f:X_1\times\cdots\times X_n\to\mathbb{R}$, we have $f:X^n\to\mathbb{R}$, we have

$$n|\nabla_1 f(x)| = \sum_{i=1}^n |\nabla_1^{(i)} f(x)|,$$

where

$$|\nabla_1^{(i)} f(x)| = \sum_{y_i\in X_i} |f(x_1,\ldots,x_n)-f(x_1,\ldots,x_{i-1},y_i,x_{i+1},\ldots,x_n)|K_i(x_i,y_i).$$

In turn this implies

$$nE_{\pi^n}|\nabla_1 f| = E_{\pi^n}\sum_{i=1}^n |\nabla_1^{(i)} f|. \tag{6.2}$$

Now

$$\begin{aligned}
\tilde{h}_1(K^n) &= \min_{0<\pi^n(A)<1} \frac{E_{\pi^n}|\nabla_1 \mathbf{1}_A|}{\mathrm{Var}_{\pi^n}(\mathbf{1}_A)} \\
&\geq \min_{0<\pi^n(A)<1} \frac{E_{\pi^n}|\nabla_1 \mathbf{1}_A|}{E_{\pi^n}\sum_{i=1}^n \mathrm{Var}_{\pi_i}\mathbf{1}_A} \\
&\geq \big(\min_{1\leq i\leq n} \tilde{h}_1(K_i)\big) \min_{0<\pi^n(A)<1} \frac{E_{\pi^n}|\nabla_1 \mathbf{1}_A|}{E_{\pi^n}\sum_{i=1}^n |\nabla_1^{(i)} \mathbf{1}_A|} \\
&= \frac{1}{n} \min_{1\leq i\leq n} \tilde{h}_1(K_i),
\end{aligned}$$

using (6.2) and the well known tensorizing property of variance (e.g., see the references in [17] or try a proof by induction): for any $f : X^n \to \mathbb{R}$

$$\mathrm{Var}_{\pi^n} f \leq E_{\pi^n} \sum_{i=1}^n \mathrm{Var}_{\pi_i} f,$$

where $\mathrm{Var}_{\pi_i} f$ is the variance with respect the "$i$th coordinate". ∎

Clearly, the approach described above improves the absolute constant in our product theorem, but is very specific to the case $p=1$. In fact, to better delineate its domain, we note that (6.1) implies that for any $f : X^n \to \mathbb{R}$ with $|f - m(f)| \leq 1$,

$$\mathrm{Var}_{\pi^n}(f) \leq 96 E_{\pi^n} \min\left(\frac{n|\nabla_1 f|}{(\min_{1\leq i\leq n} h_1(K_i))^2}, \frac{n^2|\nabla_1 f|^2}{(\min_{1\leq i\leq n} h_1(K_i))^2}\right), \tag{6.3}$$

where $m(f)$ and $Var_{\pi^n}(f)$ are respectively a median and the variance of $f$ with respect to $\pi^n$.

Indeed, applying (3.3) or (3.6) to $(X^n, \pi^n)$ with $N$ as in the proof of Theorem 3.1, leads to

$$E_{\pi^n} N(f - m(f)) \leq E_{\pi^n} N\left(\frac{4}{h_1(K^n)}|\nabla_1 f|\right). \tag{6.4}$$

for any $f$ on $X^n$. Now, if $0 \leq f \leq 1$, then $|f - m(f)| \leq 1$ and so $N(f - m(f)) \geq (f - m(f))^2/3$, therefore (6.4) and (6.1) give

$$\begin{aligned}
\frac{1}{3}\mathrm{Var}_{\pi^n}(f) &\leq E_{\pi^n}\sqrt{1 + \frac{16|\nabla_1 f|^2}{(h_1(K^n))^2}} - 1 \\
&\leq E_{\pi^n}\sqrt{1 + \frac{64 n^2|\nabla_1 f|^2}{(\min_{1\leq i\leq n} h_1(K_i))^2}} - 1.
\end{aligned}$$

Now, for all $x\in\mathbf{R}$, $\sqrt{1+4x^2}-1\le 2\min(|x|,x^2)$ and this leads to (6.3). (For $1<p\le 2$, a result similar to (6.3) also follows by replacing $\min_{1\le i\le n} h_1(K_i)/n$ by $k_p^+(K^n)$.)

Finally, we see that applying (6.3) to indicator functions of sets gives (6.1) *up to an absolute multiplicative constant.*

• The constant $1/2\sqrt{6}$ (in (4.1) or in (5.1)) is not optimal (improved constants which have no reason to be optimal either, appear in [24]). Since this work was conceived, there has been various sharpening of the constants in the following isoperimetric inequalities (introduced by Talagrand [25] for the discrete cube, viewed as a graph). Once again, let $\mu_r^n$ be the product measure on $\{0,1\}^n$, with $\mu_r$ the Bernoulli measure given by $\mu(1)=r$ and $\mu(0)=1-r$. For this special case and with $p=2$, Talagrand showed (refining a result of Margulis [22]) that there exist positive constants $C_r$ and $C_r'$ such that, for each subset $A$ of $\{0,1\}^n$,

$$E_{\mu_r^n}\nabla_2^+ 1_A \ge C_r J((\mu_r^n(A))(1-\mu_r^n(A))),$$

where $J(t)=t\sqrt{\log(1/t)}$, $t\ge 0$, and

$$E_{\mu_r^n}\nabla_2^+ 1_A \ge C_r'(\mu_r^n(A))(1-\mu_r^n(A)).$$

Refining these results further, Bobkov and Götze [3] have shown that the optimal constants can be chosen as $C_r=\Theta\left(1/\sqrt{\log(1/r)}\right)$, for $r$ small, and $C_r'=1$. Further improvements and sharpening of the constants are presented by Tillich and Zémor [26].

• The Cartesian product results presented above can be "generalized" to the non-uniform case, replacing $1/n$ by $p_i>0$ with $\sum_{i=1}^n p_i=1$. The inductive proof technique also extends to *any definition of a product* of Markov chains or graphs, (wherein the stationary measure of the product is the product of the individual stationary measures) as long as the gradients satisfy the inequality stated in Lemma 2.1 or a similar one, e.g., without the multiplicative $n$ in the above and a corresponding generalized Cheeger's inequality holds. However, at this point, we cannot even settle the case $K^n(x,y)=\prod_{i=1}^n K_i(x_i,y_i)$, where $K_i$ is a Markov kernel on $X_i$, for $i=1,\dots,n$, and $x,y\in X_1\times\cdots\times X_n$. Indeed, it is easy to show that $h_1(K^n)\le\min_j h_1(K_j)$. Nevertheless the appropriate lower bound, which should be of the same order (barring degeneracies such as *periodicity* and *reducibility*), eludes us. For this notion of a product chain, and for a new isoperimetric constant $\hat{h}$ (which generalizes $h_1$), Mark Jerrum has recently shown that $\hat{h}(K^2)\ge\frac{1}{2}\min(\hat{h}(K_1),\hat{h}(K_2))$ (see [10] for details).

• The finiteness of the state space is often unimportant. One can replace in many of our results finite sums by infinite ones or by integrals over the state space. Then combining the proofs presented here with the ones given for arbitrary (probability) measures (on metric spaces) given in [5] would give the corresponding results.

• Finally we mention that combining the above techniques with the ones developed in [20] will give similar results for (continuous time) Markovian jump processes or Markov processes with killing.

• Shortly prior to sending our manuscript to the publisher, Ben Morris informed us that, as a by-product of his recent results with Yuval Peres on "evolving sets and mixing", he could verify, up to an additional logarithmic factor, our conjecture relating the spectral gap to $h_2^+$. In particular, in the case of a simple (but modified with a self-loop probability of 1/2) random walk on a $d$-regular graph, their result implies that $\lambda \geq C((h_2^+)^2/\log d)$, for $C > 0$ an absolute constant.

## References


[1] D. Aldous and J. Fill: *Reversible Markov chains and random walks on graphs*, Book in preparation.

[2] N. Alon and V. D. Milman: $\lambda_1$, Isoperimetric inequalities for graphs, and superconcentrators, *J. Combinatorial Theory, Series B* **38** (1985), 73–88.

[3] S. G. Bobkov and F. Götze: Discrete isoperimetric and Poincaré-type inequalities, *Probab. Theory & Related Fields* **114(2)** (1999), 245–277.

[4] S. G. Bobkov and C. Houdré: Some connections between isoperimetry and Sobolev type inequalities, *Mem. Amer. Math. Soc.* **129** (1997), pp. viii+111.

[5] S. G. Bobkov and C. Houdré: Isoperimetric constants for product probability measures, *Ann. Probab.* **25** (1997), 184–205.

[6] S. G. Bobkov, C. Houdré and P. Tetali: $\lambda_\infty$, vertex isoperimetry and concentration, *Combinatorica* **20(2)** (2000), 153–172.

[7] B. Bollobás and I. Leader: Compressions and isoperimetric inequalities, *J. Combin. Theory Ser. A* **56** (1991), 47–62.

[8] B. Bollobás and I. Leader: Isoperimetric inequalities and fractional set systems, *J. Combin. Theory Ser. A* **56** (1991), 63–74.

[9] B. Bollobás and I. Leader: Edge-isoperimetric inequalities in the grid, *Combinatorica* **11** (1991), 299–314.

[10] C. Cooper, M. Dyer, A. Frieze and R. Rue: Mixing properties of the Swendsen–Wang process on the complete graph and narrow grids, *J. Math. Phys.* **41** (2000), 1499–1527.

[11] P. Diaconis and L. Saloff-Coste: Logarithmic Sobolev inequalities and finite Markov chains, *Ann. of Appl. Prob.* **6** (1996), 695–750.

[12] P. Diaconis and D. Stroock: Geometric bounds for eigenvalues of Markov chains, *Ann. of Appl. Prob.* **1** (1991), 36–61.

[13] J. A. Fill: Eigenvalue bounds on convergence to stationarity for nonreversible Markov chains, with an application to the exclusion process, *Ann. of Appl. Prob.* **1** (1991), 62–87.
[14] L. Harper: Optimal numberings and isoperimetric problems on graphs, *J. Combinatorial Theory* **1** (1966), 385–393.
[15] C. Houdré: Mixed and isoperimetric estimates on the log-Sobolev constants of graphs and Markov chains. *Combinatorica* **21(4)** (2001), 489–513.
[16] C. Houdré and P. Tetali: Isoperimetric constants for product Markov kernels and graph products, Manuscript (1996).
[17] C. Houdré and P. Tetali: Concentration of measure for products of Markov chains via functional inequalities, *Combinat. Probab. and Computing* **10** (2001), 1–28.
[18] M. Jerrum and A. Sinclair: Approximate counting, uniform generation, and rapidly mixing Markov chains, *Information and Computation* **82** (1989), 93–133.
[19] N. Kahale: A semidefinite bound for mixing rates of Markov chains, in *Proceedings of the Workshop on Randomized Algorithms and Computation* (Berkeley, CA, 1995); also *Random Structures & Algorithms* **11** (1997), 299–313.
[20] G. Lawler and A. Sokal: Bounds on the $L^2$ spectrum for Markov chains and Markov processes: A generalization of Cheeger's inequality, *Trans. Amer. Math. Soc.* **309** (1988), 557–580.
[21] L. Lovász: *Combinatorial Problems and Exercises*, 1993 (second edition), North-Holland, Elsevier Science Publishers (Amsterdam) and Akadémiai Kiadó (Budapest).
[22] G. A. Margulis: Probabilistic characteristics of graphs with large connectivity, *Problemy Peredachi Informatsii* **10** (1974), 101–108; English translation: *Problems Info. Transmission* **10** (1977), 174–179, Plenum Press, New York.
[23] B. Mohar: Isoperimetric numbers of graphs, *J. of Comb. Theory, Series B* **47** (1989), 274–291.
[24] T. Stoyanov: Doctoral Dissertation, School of Mathematics, Georgia Institute of Technology, Atlanta, GA (2001).
[25] M. Talagrand: Isoperimetry, logarithmic Sobolev inequalities on the discrete cube, and Margulis' graph connectivity theorem, *Geometric and Functional Analysis* **3** (1993), 295–314.
[26] J. P. Tillich and G. Zémor: Discrete isoperimetric inequalities and the probability of a decoding error, *Combinat. Probab. and Computing* **9** (2000), 465–479.

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